SHARP ASYMPTOTICS FOR KPP PULSATING FRONT SPEED-UP AND DIFFUSION ENHANCEMENT BY FLOWS

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ABSTRACT. We study KPP pulsating front speed-up and effective diffusivity enhancement by general periodic incompressible flows. We prove the existence of and determine the limits $c^*(A)/A$ and $D(A)/A^2$ as $A \to \infty$, where $c^*(A)$ is the minimal front speed and $D(A)$ the effective diffusivity.

1. Introduction

We study reaction-diffusion fronts in the presence of strong incompressible flows. We consider the PDE

$$T_t + Au \cdot \nabla T = \Delta T + f(T) \tag{1.1}$$

on $\mathbb{R}^n$, with $T(t,x) \in [0,1]$ the normalized temperature of a premixed combustible gas. The non-linear reaction rate f is of Kolmogorov-Petrovskii-Piskunov (KPP) type [11]:

$$\begin{gathered} f \in C^{1,\varepsilon}([0,1]), \\ f(0) = f(1) = 0 \text{ and } f \text{ is non-increasing on } (1-\varepsilon, 1) \text{ for some } \varepsilon > 0, \\ 0 < f(s) \le sf'(0) \text{ for } s \in (0,1). \end{gathered} \tag{1.2}$$

The 1-periodic flow $u : \mathbb{T}^n \to \mathbb{R}^n$ satisfies

$$u \in C^{1,\varepsilon}(\mathbb{T}^n), \qquad \nabla \cdot u \equiv 0, \qquad \int_{\mathbb{T}^n} u\, dx = 0. \tag{1.3}$$

That is, u is incompressible and mean-zero.

The number $A \in \mathbb{R}$ is the flow amplitude. We will consider the case of strong flows (i.e., large A) and their influence on the speed of propagation of pulsating fronts for (1.1). This problem has recently seen increased activity and has been addressed by various authors — see, e.g., [1, 2, 5, 7, 9, 10, 13, 14].

A *pulsating front* in the direction $e \in \mathbb{R}^n$, $|e| = 1$, is a solution of (1.1) of the form $T(t,x) = U(x \cdot e - ct, x)$, with c the front speed, and U 1-periodic in x and such that

$$\lim_{s \to -\infty} U(s,x) = 1,$$
$$\lim_{s \to +\infty} U(s,x) = 0,$$

uniformly in x. It is well known [4] that in the KPP case there is $c_e^*(A)$, called the *minimal pulsating front speed*, such that pulsating fronts exist precisely for $c \ge c_e^*(A)$ (we suppress the

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u and f dependence in our notation). We note that $c_e^*(A)$ also determines the propagation speed of solutions to the Cauchy problem with general compactly supported initial data [4, 15].

Mixing by flows (coupled to diffusion) typically increases the speed of pulsating fronts for (1.1). The minimal front speed $c_e^*(A)$ can grow at most linearly with A [5] and does so for shear (unidirectional) flows [1, 2, 7, 9]

$$u(x) = (\alpha(x'), 0, \dots, 0) \quad (x' = (x_2, \dots, x_n)) \qquad \text{and} \qquad e = (1, 0, \dots, 0). \tag{1.4}$$

The same is true for so-called *percolating* flows which possess infinite channels [7], contrasting with the case of *cellular* flows when, at least in two dimensions, $c_e^*(A) = O(A^{1/4})$ [1, 7, 9, 13] (see also [14] for a three-dimensional example).

We are interested here in all flows which maximally (i.e., linearly) enhance the minimal front speed for (1.1) and our goal is to determine the asymptotic rate of this front speed-up — to prove the existence and evaluate the limit of $c_e^*(A)/A$ as $A \to \infty$. For shear flows, this limit has been known to exist [2] and has been determined in [9], but both problems have been open in general.

We thus consider general periodic flows (1.3) and let

$$\mathcal{I} \equiv \left\{ w \in H^1(\mathbb{T}^n) \;\middle|\; \operatorname{Im} w = 0 \text{ and } u \cdot \nabla w = 0 \right\} \tag{1.5}$$

be the set of *real-valued first integrals* of the flow u. We then have the following main result.

Theorem 1.1. *If u and f satisfy* (1.2) *and* (1.3) *and $|e| = 1$, then*

$$\lim_{A\to\infty} \frac{c_e^*(A)}{A} = \sup_{\substack{w\in\mathcal{I} \\ \|\nabla w\|_2^2 \le f'(0)\|w\|_2^2}} \frac{\int_{\mathbb{T}^n} (u\cdot e) w^2 \, dx}{\|w\|_2^2}. \tag{1.6}$$

In particular, the limit exists. Moreover,

$$\lim_{f'(0)\to 0} \lim_{A\to\infty} \frac{c_e^*(A)}{2\sqrt{f'(0)A}} = \sup_{w\in\mathcal{I}} \frac{\int_{\mathbb{T}^n} (u\cdot e) w \, dx}{\|\nabla w\|_2}, \tag{1.7}$$

$$\lim_{f'(0)\to \infty} \lim_{A\to\infty} \frac{c_e^*(A)}{A} = \sup_{w\in\mathcal{I}} \frac{\int_{\mathbb{T}^n} (u\cdot e) w^2 \, dx}{\|w\|_2^2} \le \max_{x\in\mathbb{T}^n} \{u(x)\cdot e\}. \tag{1.8}$$

Remarks. 1. Inequality "$\ge$" in (1.6) (with $\liminf_{A\to\infty}$ in place of $\lim_{A\to\infty}$) has been proved in [5], and [9] showed equality in the case of shear flows (1.4).

2. (1.8) already appeared in [5], with either $\liminf_{A\to\infty}$ or $\limsup_{A\to\infty}$ in place of $\lim_{A\to\infty}$. For shear flows (1.4) the inequality becomes an equality [9] due to (1.6) and continuity of u.

3. Notice that (1.6) (for any $f'(0)$) is positive precisely when there exists $w \in \mathcal{I}$ such that $\int_{\mathbb{T}^n} (u\cdot e) w \, dx \ne 0$ (take $1 \pm \varepsilon w$ in (1.6)). This is also the condition for positivity of (1.7) and (1.12) below.

4. The result extends directly to the more general case of x-dependent and 1-periodic reaction and second-order term (see Theorem 3.2). We perform the proof in the simpler setting above for the sake of transparency.

It has been shown in [13, 14] that, at least in two dimensions, there is a close relationship between the minimal front speeds for (1.1) and the effective diffusivity in the homogenization theory for the related advection-diffusion problem

$$\Phi_t + Au \cdot \nabla\Phi = \Delta\Phi. \tag{1.9}$$

As is well known, the long-time behavior of solutions to (1.9) is governed by the effective diffusion equation

$$\Psi_t = \sum_{i,j=1}^{n} \sigma_{ij}(A) \frac{\partial^2 \Psi}{\partial x_i \partial x_j}.$$

Here $\sigma(A)$ is a constant effective diffusivity matrix. If $e \in \mathbb{R}^n$ and we let $\chi_{e,A}$ be the mean-zero solution of

$$-\Delta\chi_{e,A} + Au \cdot \nabla\chi_{e,A} = Au \cdot e \tag{1.10}$$

on $\mathbb{T}^n$, then $\sigma(A)$ is given by

$$e \cdot \sigma(A)e' = \int_{\mathbb{T}^n} (\nabla\chi_{e,A} + e) \cdot (\nabla\chi_{e',A} + e')dx = e \cdot e' + \int_{\mathbb{T}^n} \nabla\chi_{e,A} \cdot \nabla\chi_{e',A} dx.$$

The *effective diffusivity* for (1.9) in the direction $e \in \mathbb{R}^n$, $|e| = 1$, is now

$$D_e(A) \equiv e \cdot \sigma(A)e = 1 + \|\nabla\chi_{e,A}\|_2^2. \tag{1.11}$$

Again, mixing by flows enhances the effective diffusivity. It is easy to show that $D_e(A)$ can grow at most quadratically with A, and flows that achieve this are said to *maximally enhance diffusion* (see [6, 8, 12] and references therein). It turns out that our method applies to the problem of determining the asymptotic rate of this enhancement as well, and we find the limit $D_e(A)/A^2$ as $A \to \infty$ for general periodic flows. To the best of the author's knowledge, existence of this limit has not been known before.

Theorem 1.2. *If u satisfies* (1.3) *and $|e| = 1$, then*

$$\lim_{A\to\infty} \frac{D_e(A)}{A^2} = \sup_{w\in\mathcal{I}} \left(\frac{\int_{\mathbb{T}^n} (u \cdot e) w \, dx}{\|\nabla w\|_2} \right)^2. \tag{1.12}$$

In particular, the limit exists. Moreover, there is $w_0 \in \mathcal{I}$ which is a maximizer of (1.12) *and $\chi_{e,A}/A \to w_0$ in $H^1(\mathbb{T}^n)$.*

Remarks. 1. It follows that the left hand side of (1.12) is the square of the left hand side of (1.7). This has been established in two dimensions by Ryzhik and the author [14], even without the $A \to \infty$ limit (see also [13]).

2. We show that if (1.12) is positive, then the maximizers are precisely $w = aw_0 + b$ with $a, b \in \mathbb{R}$, $a \neq 0$.

3. If one considers the small diffusion problem $\phi_t = \varepsilon\Delta\phi + u \cdot \nabla\phi$ instead of (1.9), then the corresponding effective diffusivity satisfies $\tilde{D}_e(\varepsilon) = \varepsilon D_e(\varepsilon^{-1})$. Hence the limit $\lim_{\varepsilon\to 0} \varepsilon\tilde{D}_e(\varepsilon)$ also equals (1.12).

4. Again, there is a straightforward extension to the case of x-dependent second order term and even non-mean-zero flows (see Theorem 2.1).

We prove Theorem 1.2 in Section 2 and Theorem 1.1 in Section 3. The generalizations to the case of x-dependent second-order and reaction terms are Theorems 2.1 and 3.2 below.

2. Effective Diffusivity Enhancement

Proof of Theorem 1.2. Let $\psi_A \equiv \chi_{e,A}/A$, so that

$$-\Delta\psi_A + Au\cdot\nabla\psi_A = u\cdot e \tag{2.1}$$

Multiplying this by ψ_A and integrating over $\mathbb{T}^n$ we obtain using incompressibility of the flow,

$$\|\nabla\psi_A\|_2^2 = \int_{\mathbb{T}^n} (u\cdot e)\psi_A dx \le \|u\cdot e\|_2\|\psi_A\|_2. \tag{2.2}$$

Poincaré inequality

$$\|w\|_2 \le C\|\nabla w\|_2 \tag{2.3}$$

for some $C<\infty$ and any mean-zero w then yields

$$\|\psi_A\|_{H^1} \le C\|u\cdot e\|_2. \tag{2.4}$$

It also follows from (2.2) that

$$\frac{D_e(A)}{A^2} = \frac{1}{A^2} + \|\nabla\psi_A\|_2^2 = \frac{1}{A^2} + \int_{\mathbb{T}^n}(u\cdot e)\psi_A dx. \tag{2.5}$$

Since $\|\psi_A\|_{H^1}$ is uniformly bounded, there is a sequence $A_k\to\infty$ such that ψ_{A_k} converges to some $w_0\in H^1(\mathbb{T}^n)$, weakly in $H^1(\mathbb{T}^n)$ and strongly in $L^2(\mathbb{T}^n)$. Then $\Delta\psi_{A_k}\to\Delta w_0$ and $\nabla\psi_{A_k}\to\nabla w_0$ in the sense of distributions and (2.1) divided by A_k implies

$$u\cdot\nabla w_0 = 0 \tag{2.6}$$

in the sense of distributions. Since $w_0\in H^1(\mathbb{T}^n)$, this equality holds almost everywhere and $w_0\in\mathcal{I}$. We also have

$$\|\nabla w_0\|_2^2 \le \limsup_{k\to\infty}\|\nabla\psi_{A_k}\|_2^2 = \int_{\mathbb{T}^n}(u\cdot e)w_0 dx = \int_{\mathbb{T}^n}\nabla\psi_A\nabla w_0 dx \le \|\nabla\psi_A\|_2\|\nabla w_0\|_2$$

where we used (2.2) in the second step, and (2.1) multiplied by w_0 and integrated over $\mathbb{T}^n$ (together with (2.6)) in the third step. Thus

$$\|\nabla w_0\|_2 \le \|\nabla\psi_A\|_2 \tag{2.7}$$

as well as

$$\limsup_{k\to\infty}\|\nabla\psi_{A_k}\|_2 \le \|\nabla w_0\|_2.$$

These give

$$\lim_{k\to\infty}\|\nabla\psi_{A_k}\|_2 = \|\nabla w_0\|_2,$$

which turns the weak H^1-convergence into a strong one:

$$\psi_{A_k}\to w_0 \qquad \text{in } H^1(\mathbb{T}^n). \tag{2.8}$$

Let us assume $w_0 \not\equiv 0$. Then $\nabla w_0 \not\equiv 0$ because each ψ_A is mean-zero. From (2.5) and (2.8),

$$\lim_{k\to\infty} \frac{D_e(A_k)}{A_k^2} = \|\nabla w_0\|_2^2 = \int_{\mathbb{T}^n} (u\cdot e) w_0\, dx = \left(\frac{\int_{\mathbb{T}^n} (u\cdot e) w_0\, dx}{\|\nabla w_0\|_2} \right)^2. \tag{2.9}$$

Pick an arbitrary non-constant $w \in \mathcal{I}$. If we multiply (2.1) by w and integrate, we obtain

$$\left| \int_{\mathbb{T}^n} (u\cdot e) w\, dx \right| = \left| \lim_{k\to\infty} \int_{\mathbb{T}^n} \nabla\psi_{A_k} \nabla w\, dx \right| = \left| \int_{\mathbb{T}^n} \nabla w_0 \nabla w\, dx \right| \le \|\nabla w_0\|_2 \|\nabla w\|_2. \tag{2.10}$$

Hence

$$\left(\frac{\int_{\mathbb{T}^n} (u\cdot e) w\, dx}{\|\nabla w\|_2} \right)^2 \le \|\nabla w_0\|_2^2 = \lim_{k\to\infty} \frac{D_e(A_k)}{A_k^2},$$

with equality precisely when ∇w is a multiple of ∇w_0 (and so $w = aw_0 + b$). This also means that w_0 is a maximizer for (1.12).

If now $B_k \to \infty$ is any sequence, then as above we can find a subsequence (which we again call B_k) such that $\psi_{B_k} \to w_1 \in \mathcal{I}$. But then w_1 must also maximize (1.12), thus $w_1 = aw_0 + b$. Moreover, $b = 0$ because ψ_A are mean-zero, and (2.9) with B_k in place of A_k forces $a = 1$. Hence $\psi_A \to w_0$ in $H^1(\mathbb{T}^n)$ and (1.12) follows.

Finally, if $w_0 \equiv 0$ is the only limit point of ψ_A, then $\psi_A \to 0$ in $H^1(\mathbb{T}^n)$, and (1.12) follows from (2.5) and (2.10). $\square$

Notice that (2.5), (2.7), and (2.9) show that $D_e(A) \ge 1 + \delta A^2$, where δ is the limit in (1.12).

We also note that in the special case of shear flows $u(x) = (\alpha(x'), 0, \dots, 0)$ equation (1.10) becomes

$$-\Delta_{x'} \chi_{e,A} = Ae_1 \alpha(x')$$

with $\chi_{e,A}(x) = \chi_{e,A}(x')$. Hence $\chi_{e,A} = Ae_1 \nabla_{x'} (-\Delta_{x'})^{-1} \alpha$ and the limit in (1.12) equals $|e_1| \|\nabla_{x'} (-\Delta_{x'})^{-1} \alpha\|_2^2$. This can be found, e.g., in [8, Lemma 7.3].

As mentioned above, the result easily extends to the case of x-dependent second order term and a non-mean-zero flow. We consider

$$\Phi_t + Au\cdot\nabla\Phi = \nabla\cdot(a\nabla\Phi) \tag{2.11}$$

instead of (1.9) with 1-periodic and real symmetric uniformly elliptic matrix a and 1-periodic flow u such that

$$a \in C^2(\mathbb{T}^n), \qquad u \in C^{1,\alpha}(\mathbb{T}^n), \qquad \nabla\cdot u \equiv 0, \qquad \bar{u} \equiv \int_{\mathbb{T}^n} u\, dx. \tag{2.12}$$

Then (1.10) and (1.11) are replaced by

$$-\nabla\cdot(a\nabla\chi_{e,A}) + Au\cdot\nabla\chi_{e,A} = A(u-\bar{u})\cdot e,$$
$$D_e(A) \equiv |||\nabla\chi_{e,A} + e|||_2^2,$$

with $|||w|||_2^2 \equiv \int_{\mathbb{T}^n} \nabla w \cdot (a\nabla w)\, dx$. If we define

$$\mathcal{I}_0 \equiv \left\{ w \in \mathcal{I} \,\middle|\, \int_{\mathbb{T}^n} w\, dx = 0 \right\},$$

then we have

Theorem 2.1. *If a and u satisfy (2.12) and $|e| = 1$, then*

$$\lim_{A\to\infty} \frac{D_e(A)}{A^2} = \sup_{w\in\mathcal{I}_0} \left(\frac{\int_{\mathbb{T}^n} (u\cdot e) w\, dx}{|||\nabla w|||_2} \right)^2. \tag{2.13}$$

In particular, the limit exists. Moreover, there is $w_0 \in \mathcal{I}_0$ which is a maximizer of (2.13) and $\chi_{e,A}/A \to w_0$ in $H^1(\mathbb{T}^n)$.

3. KPP Front Speed-up

In this section we prove Theorem 1.1. We start with an auxiliary lemma. Let us define

$$\kappa_e(\lambda) \equiv \sup_{w\in\mathcal{I}} \left\{ \frac{\lambda \int_{T^n} (u\cdot e) w^2\, dx - \|\nabla w\|_2^2}{\|w\|_2^2} \right\}. \tag{3.1}$$

Note that $\kappa_e(\lambda)$ must be convex as it is a supremum of linear functions. Also, $\kappa_e(\lambda) \geq 0$ because $w \equiv 1 \in \mathcal{I}$.

Lemma 3.1. *Assume the setting of Theorem 1.1. Then for each $\lambda > 0$, the supremum in (3.1) is attained, the maximizer is unique up to multiplication, and*

$$\lim_{A\to\infty} \frac{c_e^*(A)}{A} = \inf_{\lambda>0} \frac{f'(0) + \kappa_e(\lambda)}{\lambda}. \tag{3.2}$$

Proof. It has been shown in [4] that the minimal front speed $c_e^*(A)$ can be computed using the variational principle

$$c_e^*(A) = \inf_{\lambda>0} \frac{f'(0) + \lambda^2 + \kappa(\lambda; A)}{\lambda}. \tag{3.3}$$

Here $\kappa(\lambda; A)$ is the unique eigenvalue of the problem

$$\Delta\varphi - Au\cdot\nabla\varphi - 2\lambda e\cdot\nabla\varphi + \lambda A u\cdot e\varphi = \kappa(\lambda; A)\varphi, \qquad \varphi > 0 \tag{3.4}$$

on $\mathbb{T}^n$, with a unique normalized eigenfunction $\varphi_A(x;\lambda)$. Moreover, the function

$$\mu(\lambda; A) \equiv \lambda^2 + \kappa(\lambda; A)$$

is monotonically increasing and convex in $\lambda \geq 0$, with $\mu(0; A) = 0$ (see [3, 13]).

We now rewrite (3.3) and (3.4) as

$$\frac{c_e^*(A)}{A} = \inf_{\lambda>0} \frac{f'(0) + (\lambda/A)^2 + \kappa(\lambda/A; A)}{\lambda}. \tag{3.5}$$

and

$$\Delta\varphi_A - Au\cdot\nabla\varphi_A - \frac{2\lambda}{A} e\cdot\nabla\varphi_A + \lambda u\cdot e\varphi_A = \kappa(\lambda/A; A)\varphi_A, \qquad \varphi_A > 0. \tag{3.6}$$

We multiply (3.6) by φ_A^{-1} and integrate to obtain (using incompressibility of u)

$$0 \leq \|\nabla \ln\varphi_A\|_2^2 = \kappa(\lambda/A; A). \tag{3.7}$$

Similarly, multiplication by φ_A yields

$$\kappa(\lambda/A;A) + \|\nabla\varphi_A\|_2^2 = \lambda\int_{\mathbb{T}^n}(u\cdot e)\varphi_A^2 dx \le \lambda\|u\cdot e\|_\infty. \tag{3.8}$$

since $\|\varphi_A\|_2 = 1$. This again means that there is a sequence $A_k \to \infty$ such that φ_{A_k} converges to some $w_0 \in H^1(\mathbb{T}^n)$, weakly in $H^1(\mathbb{T}^n)$ and strongly in $L^2(\mathbb{T}^n)$. The convergence $\Delta\varphi_{A_k} \to \Delta w_0$ and $\nabla\varphi_{A_k} \to \nabla w_0$ in the sense of distributions, boundedness of $\kappa(\lambda/A;A)$ in A, and (3.6) divided by A then imply (2.6) and so $w_0 \in \mathcal{I}$ (note that $\|w_0\|_2 = \|\varphi_{A_k}\|_2 = 1$).

Now we multiply (3.6) by w_0 and integrate to obtain (with $o(1) = o(k^0)$ and using (3.8))

$$\begin{aligned}-\int_{\mathbb{T}^n}\nabla\varphi_{A_k}\nabla w_0 dx + \lambda\int_{\mathbb{T}^n}(u\cdot e)w_0^2 dx + o(1) &= \kappa(\lambda/A_k;A_k) + o(1)\\ &= \lambda\int_{\mathbb{T}^n}(u\cdot e)w_0^2 dx - \|\nabla\varphi_{A_k}\|_2^2 + o(1).\end{aligned}$$

Once again it follows that

$$\|\nabla w_0\|_2^2 \le \limsup_{k\to\infty}\|\nabla\varphi_{A_k}\|_2^2 \le \|\nabla w_0\|_2 \limsup_{k\to\infty}\|\nabla\varphi_{A_k}\|_2$$

and so as in Section 2,

$$\varphi_{A_k} \to w_0 \qquad \text{in } H^1(\mathbb{T}^n). \tag{3.9}$$

(3.8) then yields

$$\kappa_0 \equiv \lim_{k\to\infty}\kappa(\lambda/A_k;A_k) = \lambda\int_{\mathbb{T}^n}(u\cdot e)w_0^2 dx - \int_{\mathbb{T}^n}|\nabla w_0|^2\,dx.$$

Let $w \in \mathcal{I}\cap L^\infty(\mathbb{T}^n)$, multiply (3.6) for $A = A_k$ by w^2/φ_{A_k} and integrate to obtain (using that $\nabla\varphi_A/\varphi_A = \nabla\ln\varphi_A$ are uniformly bounded in $L^2(\mathbb{T}^n)$ by (3.7) and (3.8))

$$\begin{aligned}\kappa_0\|w\|_2^2 &= \lambda\int_{\mathbb{T}^n}(u\cdot e)w^2 dx + \lim_{k\to\infty}\int_{\mathbb{T}^n}\left|\frac{\nabla\varphi_{A_k}}{\varphi_{A_k}}\right|^2 w^2 - 2\frac{\nabla\varphi_{A_k}}{\varphi_{A_k}}w\nabla w - \frac{2\lambda}{A_k}e\cdot\frac{\nabla\varphi_{A_k}}{\varphi_{A_k}}w^2\,dx\\ &\ge \lambda\int_{\mathbb{T}^n}(u\cdot e)w^2 dx - \int_{\mathbb{T}^n}|\nabla w|^2\,dx.\end{aligned} \tag{3.10}$$

Since each $w \in \mathcal{I}$ is the H^1-limit of $w_N(x) \equiv \operatorname{sgn}(w(x))\min\{|w(x)|, N\} \in \mathcal{I}\cap L^\infty(\mathbb{T}^n)$, this inequality extends to all $w \in \mathcal{I}$. Hence $\kappa_0 = \kappa_e(\lambda)$ from (3.1), and w_0 is a maximizer for (3.1) (because $\|w_0\|_2 = 1$). Moreover, if $B_k \to \infty$ is any sequence with

$$\lim_{k\to\infty}\kappa(\lambda/B_k;B_k) \equiv \kappa_1,$$

then repeating the above argument we find that there must be a subsequence (which we again call B_k) such that $\varphi_{B_k} \to w_1 \in \mathcal{I}$ in $H^1(\mathbb{T}^n)$, $\|w_1\|_2 = 1$. But then as before,

$$\lambda\int_{\mathbb{T}^n}(u\cdot e)w_1^2 dx - \int_{\mathbb{T}^n}|\nabla w_1|^2\,dx = \kappa_1 \ge \frac{\lambda\int_{\mathbb{T}^n}(u\cdot e)w^2 dx - \int_{\mathbb{T}^n}|\nabla w|^2\,dx}{\|w\|_2^2}$$

for any $w \in \mathcal{I}$. Taking $w = w_0$ we obtain $\kappa_1 = \kappa_e(\lambda)$, and so

$$\kappa_e(\lambda) = \lim_{A\to\infty}\kappa(\lambda/A;A) = \lambda\int_{\mathbb{T}^n}(u\cdot e)w_0^2 dx - \int_{\mathbb{T}^n}|\nabla w_0|^2\,dx. \tag{3.11}$$

The function $\kappa_e(\lambda)$ is convex, monotonically increasing, and non-negative, as it is the pointwise limit of functions $\mu(\lambda/A;A) = (\lambda/A)^2 + \kappa(\lambda/A;A)$ which have the same properties. This also implies that the convergence in (3.11) is uniform on each bounded interval of λ. We then have

$$\lim_{A\to\infty} \inf_{\lambda>0} \frac{f'(0)+\mu(\lambda/A;A)}{\lambda} = \inf_{\lambda>0} \frac{f'(0)+\kappa_e(\lambda)}{\lambda}$$

($\le$ is immediate, whereas $\ge$ uses convexity of $\mu(\lambda/A;A)$ once more). This proves (3.2).

We are left with showing that any maximizer of (3.1) is a multiple of w_0. Denote $\varphi_k \equiv \varphi_{A_k}$ and notice that (3.9) shows that (after passing to a subsequence — we will repeat this without mentioning it below), $\nabla\varphi_k(x) \to \nabla w_0(x)$ and $\varphi_k(x) \to w_0(x)$ for a.e. x. Next (3.7) and (2.3) imply that if c_k is the average of $\ln\varphi_k$, then $\ln\varphi_k - c_k \to \omega$ strongly in L^2 and weakly in H^1. But then $\ln\varphi_k(x) - c_k \to \omega(x)$ for a.e. x. Since $\ln\varphi_k(x) \to \ln w_0(x)$ for a.e. x, it follows that $c_k \to c$ and $\omega = \ln w_0 - c$. We thus obtain $\ln w_0 \in H^1$ which means $w_0(x) \neq 0$ for a.e. x, and so for a.e. x,

$$\frac{\nabla\varphi_k(x)}{\varphi_k(x)} \to \frac{\nabla w_0(x)}{w_0(x)}. \tag{3.12}$$

Let now $w \not\equiv 0$ be a maximizer of (3.1) and let us first assume $w \ge 0$ almost everywhere. Then (3.10) for w_N and $w_N \to w$ in H^1 show

$$\lim_{N\to\infty} \limsup_{k\to\infty} \left\| \frac{\nabla\varphi_k}{\varphi_k} w_N - \nabla w_N \right\|_2 = 0.$$

But then (3.12) and pointwise convergence of w_N and ∇w_N to w and ∇w, respectively, give for a.e. x,

$$\frac{\nabla w_0(x)}{w_0(x)} w(x) = \nabla w(x).$$

We now let $w_\varepsilon(x) \equiv \max\{w(x), \varepsilon\}$ so that $\ln w_\varepsilon \in H^1$ and

$$\nabla \ln w_\varepsilon(x) = \begin{cases} \nabla \ln w_0(x) & w_\varepsilon(x) > \varepsilon, \\ 0 & w_\varepsilon(x) = \varepsilon. \end{cases}$$

This and $\ln w_0 \in H^1$ means that $\|\nabla \ln w_\varepsilon\|_2$ is bounded, and again we must have $\ln w_{\varepsilon_k} - c_{\varepsilon_k} \to \omega$ strongly in L^2, weakly in H^1, and pointwise almost everywhere. But $\ln w_{\varepsilon_k}(x) \to \ln w(x)$, so again $c_{\varepsilon_k} \to c$ and $\ln w \in H^1$. Hence $w(x) > 0$ for a.e. x, and so $\nabla \ln w(x) = \nabla \ln w_0(x)$ for a.e. x. This means $\ln w - \ln w_0$ is constant, that is, w is a multiple of w_0.

If w is an arbitrary maximizer of (3.1), then both $w_\pm(x) \equiv \max\{\pm w(x), 0\} \in \mathcal{I}$ must be maximizers of (3.1) (or $\equiv 0$). But then $w_\pm(x) > 0$ for a.e. x, meaning that one of them is zero while the other is a multiple of w_0. □

Proof of Theorem 1.1. Inequality "$\ge$" in (1.6) is immediate from (3.1) and (3.2). To prove the opposite inequality it is sufficient to find λ such that the unique normalized non-negative maximizer $w_0(\lambda)$ of (3.1) satisfies $\gamma(\lambda) \equiv \|\nabla w_0(\lambda)\|_2^2 = f'(0)$.

To this end notice that if $\lambda = 0$, then $w_0(\lambda) \equiv 1$ and so $\gamma(0) = 0$. Also, γ must be continuous. Indeed — let $\lambda_k \to \lambda_\infty < \infty$ and denote $w_k \equiv w_0(\lambda_k)$. Then (3.8) and (3.9) imply that w_k are uniformly bounded in H^1. Thus a subsequence (again called w_k) converges

strongly in L^2 and weakly in H^1 to some ω. Obviously $\omega \in \mathcal{I}$ as well as $\lambda_k \int (u\cdot e) w_k^2\,dx \to \lambda_\infty \int (u\cdot e)\omega^2\,dx$ and $\|\nabla\omega\|_2 \le \liminf \|\nabla w_k\|_2$. But $\|\nabla\omega\|_2 < \liminf \|\nabla w_k\|_2$ is impossible (otherwise w_k would not maximize (3.1) for large k) and so for a subsequence,

$$\kappa_e(\lambda_k) = \lambda_k \int (u\cdot e) w_k^2\,dx - \|\nabla w_k\|_2^2 \to \lambda_\infty \int (u\cdot e)\omega^2\,dx - \|\nabla\omega\|_2^2.$$

Since κ_e is continuous, this means $\omega = w_\infty$. We have thus proved that every sequence $\lambda_k \to \lambda_\infty$ has a subsequence with $\gamma(\lambda_{k_j}) \to \gamma(\lambda_\infty)$, that is, γ is continuous with $\gamma(0)=0$.

Let now $\Gamma \equiv \sup_{\lambda\ge 0}\gamma(\lambda)$. If $f'(0)\in(0,\Gamma)$, then there is $\lambda>0$ with $\gamma(\lambda)=f'(0)$ and (1.6) is proved. If, on the other hand, $\Gamma<\infty$ and $f'(0)\ge\Gamma$, then (3.2) is bounded above by

$$\liminf_{\lambda\to\infty} \frac{\lambda\int (u\cdot e) w_0(\lambda)^2\,dx - \|\nabla w_0(\lambda)\|_2^2 + f'(0)}{\lambda} = \liminf_{\lambda\to\infty} \int (u\cdot e) w_0(\lambda)^2\,dx,$$

which does not exceed the right hand side of (1.6) due to $\|\nabla w_0(\lambda)\|_2^2 \le \Gamma \le f'(0)\|w_0(\lambda)\|_2^2$.

Since (1.8) is immediate from (1.6), we are left with proving (1.7). Let us consider any $w\in\mathcal{I}$ with $\|w\|_2=1$ and $\|\nabla w\|_2^2 \le f'(0)$. Let $\bar w \equiv \int_{\mathbb{T}^n} w\,dx \in [-1,1]$ and $\omega \equiv w-\bar w$. Then $\|\omega\|_2^2 \le C\|\nabla\omega\|_2^2 \le Cf'(0)$, and so

$$1 = \int_{\mathbb{T}^n} \bar w^2 + 2\bar w\omega + \omega^2\,dx = \bar w^2 + O(\sqrt{f'(0)})$$

as $f'(0)\to 0$. Hence $\bar w = 1 + O(\sqrt{f'(0)})$ and we have

$$\int_{\mathbb{T}^n} (u\cdot e) w^2\,dx = 2\int_{\mathbb{T}^n} (u\cdot e)\omega\,dx + O(f'(0)) \le 2\sqrt{f'(0)}\frac{\int_{\mathbb{T}^n}(u\cdot e)\omega\,dx}{\|\nabla\omega\|_2} + O(f'(0))$$

with equality when $\|\nabla\omega\|_2^2 = f'(0)$. Picking first w that maximizes (1.6) and then ω that maximizes (1.7) with $\|\nabla\omega\|_2^2 = f'(0)$ (and adjusting $\bar w$ accordingly) finishes the proof. $\square$

Note that in the case of shear flows (1.4) equation (3.6) becomes

$$\Delta_{x'}\varphi + \lambda\alpha\varphi = \kappa(\lambda/A;A)\varphi, \qquad \varphi>0. \tag{3.13}$$

with $\varphi(x)=\varphi(x')$. As a result $\kappa(\lambda/A;A)=\kappa(\lambda;1)=\kappa_e(\lambda)$ and $w_0(\lambda)=\varphi$, and (3.5) shows that $c_e^*(A)/A$ is non-increasing. This has been proved in [2]. If the limit is γ, then (3.5) gives

$$\frac{c_e^*(A)}{A} - \gamma \le \frac{2\sqrt{f'(0)}}{A}$$

(which has been already observed in [9]). Here one uses convexity of κ_e and $\kappa_e(0)=0$ to show that the infimum in (3.5) is achieved at some $\lambda \le \lambda_A \equiv \sqrt{f'(0)}A$, as well as

$$\inf_{\lambda>0} \frac{f'(0)+\kappa_e(\lambda)}{\lambda} \ge \min\left\{ \inf_{\lambda\in(0,\lambda_A)} \frac{f'(0)+\kappa_e(\lambda)}{\lambda}, \frac{\kappa_e(\lambda_A)}{\lambda_A} \right\}.$$

Moreover, if the infimum in (3.2) is achieved at a finite λ, then (3.5) gives that

$$\frac{c_e^*(A)}{A} - \gamma = O(A^2).$$

This condition is satisfied for all $f'(0) < \Gamma$, where Γ is from the proof of Lemma 3.1, that is, it is the supremum over $\lambda > 0$ of the $\dot{H}^1$ norms of the principal eigenfunctions of (3.13). This is because of (3.2), the definition of κ_e, and the fact that

$$\lim_{\lambda\to\infty} \frac{\kappa_e(\lambda)}{\lambda} = \sup_{w\in\mathcal{I}} \frac{\int_{\mathbb{T}^n}(u\cdot e)w^2\,dx}{\|w\|_2^2}.$$

Finally, we note that $\Gamma < \infty$ is possible — in the shear flow case it holds when there is an open set $U \subseteq \mathbb{T}^{n-1}$ such that $\alpha(x') = \max_{\mathbb{T}^{n-1}} \alpha$ for all $x' \in U$. Then any $w \in H^1(\mathbb{T}^n)$ supported on $\mathbb{T}\times U$ and independent of x_1 belongs to $\mathcal{I}$ and maximizes (1.6) whenever $f'(0) \ge \|\nabla w\|_2^2/\|w\|_2^2$. Thus the limit in (1.6) need not be strictly increasing with $f'(0)$ (which happens precisely when $\Gamma < \infty$).

In the more general case when the second order term and the non-linearity depend on x, we consider

$$T_t + Au\cdot\nabla T = \nabla\cdot(a\nabla T) + f(x,T) \tag{3.14}$$

with a 1-periodic real symmetric uniformly elliptic matrix and u 1-periodic such that

$$a \in C^2(\mathbb{T}^n), \qquad u \in C^{1,\varepsilon}(\mathbb{T}^n), \qquad \nabla\cdot u \equiv 0, \qquad \int_{\mathbb{T}^n} u\,dx = 0. \tag{3.15}$$

The non-linearity f is 1-periodic in x and satisfies for some $\varepsilon > 0$

$$\begin{gathered} f \in C^{1,\delta}(\mathbb{T}^n\times[0,1]), \\ f(x,0) = f(x,1) = 0 \text{ and } f(x,\cdot) \text{ is non-increasing on } (1-\varepsilon,1) \text{ for each } x\in\mathbb{T}^n, \\ 0 < f(x,s) \le s f_s'(x,0) \text{ for } (s,x)\in(0,1)\times\mathbb{T}^n. \end{gathered} \tag{3.16}$$

We let $\zeta(x) \equiv f_s'(x,0) > 0$ and $\zeta_0 \equiv \int_{\mathbb{T}^n}\zeta(x)\,dx$. Equations (3.3) and (3.4) are then replaced by (see [4])

$$c_e^*(A) = \inf_{\lambda>0}\frac{\kappa(\lambda,f;A)}{\lambda},$$

$$\nabla\cdot(a\nabla\varphi) - Au\cdot\nabla\varphi - 2\lambda e\cdot a\nabla\varphi + [\lambda Au\cdot e + \zeta + \lambda^2 e\cdot ae - \lambda\nabla\cdot(ae)]\varphi = \kappa(\lambda,f;A)\varphi.$$

If we now define

$$\kappa_e(\lambda,f) \equiv \sup_{w\in\mathcal{I}}\left\{\frac{\int_{T^n}(\lambda u\cdot e+\zeta)w^2\,dx - |||\nabla w|||_2^2}{\|w\|_2^2}\right\},$$

then (3.2) becomes

$$\lim_{A\to\infty}\frac{c_e^*(A,f)}{A} = \inf_{\lambda>0}\frac{\kappa_e(\lambda,f)}{\lambda}$$

and mimicking the above proofs one obtains the following extension of Theorem 1.1.

Theorem 3.2. *If a, u, and f satisfy* (3.15) *and* (3.16) *and $|e| = 1$, then*

$$\lim_{A\to\infty}\frac{c_e^*(A)}{A} = \sup_{\substack{w\in\mathcal{I}\\ |||\nabla w|||_2^2 \le \int_{\mathbb{T}^n}\zeta w^2\,dx}} \frac{\int_{\mathbb{T}^n}(u\cdot e)w^2\,dx}{\|w\|_2^2}.$$

In particular, the limit exists. Moreover,

$$\lim_{\alpha\to 0}\lim_{A\to\infty}\frac{c_e^*(A,\alpha f)}{2\sqrt{\alpha\zeta_0}A}=\sup_{w\in\mathcal{I}}\frac{\int_{\mathbb{T}^n}(u\cdot e)w\,dx}{|||\nabla w|||_2},$$
$$\lim_{\alpha\to\infty}\lim_{A\to\infty}\frac{c_e^*(A,\alpha f)}{A}=\sup_{w\in\mathcal{I}}\frac{\int_{\mathbb{T}^n}(u\cdot e)w^2\,dx}{\|w\|_2^2}\le\max_{x\in\mathbb{T}^n}\{u(x)\cdot e\}.$$

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